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Calculation of longitudinal shear dispersivity using an N-zone model as $N \to \infty$

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A new method is presented for deriving an integral expression for calculating the large-time longitudinal shear dispersivity in laminar or turbulent two-dimensional channel flow or tube flow.

1. Introduction

There are basically four analytical methods of deriving a formula for the longitudinal dispersivity of contaminants in shear flows. The first was presented by Taylor (1953, 1954) in his pioneering work on dispersion in laminar or turbulent flows in tubes. Taylor's method was applied by Elder (1959) to turbulent open channel flow and has been extended to three-dimensional flows (Fischer *et al.* 1979; Chatwin & Sullivan 1982). The second method is the concentration-moment method which was developed by Aris (1956) and has been generalized by Brenner (1980). In the third method (Gill & Sankarasubramanian 1970) the dispersivity is obtained as a function of time. This method has also been extended to three-dimensional flows (Doshi, Daiya & Gill 1978), and has been modified by Maron (1978) and Smith (1981). The fourth approach is the probabilistic formulation of the turbulent dispersion problem which was given by Batchelor, Binnie & Phillips (1955) and has been further validated by Lumley (1972). This probabilistic approach has also been used in some laminar flows (Jimenez & Sullivan 1984).

The aim of this paper is to present a different method of deriving a formula for calculating the longitudinal dispersivity of contaminants in laminar or turbulent two-dimensional channel flow or tube flow. This is a method of zones in which the flow is divided into N zones of parallel flow, each of which is considered to be well mixed. A dispersion equation is obtained in each zone and cross-stream mixing between the zones is taken into account, leading to a system of N coupled dispersion equations. The exact longitudinal dispersivity at large times is calculated for the N-zone model and in the limit as N tends to infinity an integral expression is obtained for the dispersivity. This integral expression is in agreement with the formulas of Taylor and Aris and is an exact large-time analytical result.

The governing advective–diffusion equation for contaminant dispersion in two-dimensional open-channel flow of depth h is

$$\frac{\partial c}{\partial t} = D_x(y)\frac{\partial^2 c}{\partial x^2} - u(y)\frac{\partial c}{\partial x} + \frac{\partial}{\partial y}\left(D_y(y)\frac{\partial c}{\partial y}\right), \tag{1.1}$$

where c is contaminant concentration, $u(y)$ is the cross-sectional velocity distribution, x is longitudinal distance down the channel, y is vertical distance downward from the maximum velocity surface, D_x and D_y are the diffusivities (laminar or turbulent) in the x- and y-directions respectively.

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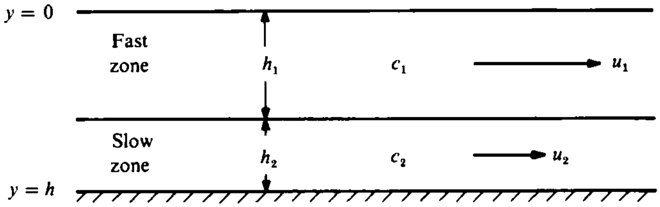

FIGURE 1. Open-channel flow showing fast and slow zones.

The starting point of the method of zones is the slow-zone model (Thacker 1976; Smith 1982; Chikwendu & Ojiakor 1985) in which the flow is divided into two zones. In the slow-zone model (see figure 1) the flow is divided into a fast zone of thickness h_1 near the middle and a slow zone of thickness h_2 near the wall. Each zone is assumed to be well mixed, with contaminant concentrations (c_1 and c_2), flow velocities (u_1 and u_2), diffusivities (D_{x1} and D_{x2}) in the fast and slow zones respectively, and a mixing coefficient b between the zones. The resulting coupled dispersion equations are

$$\partial_t c_1 = D_{x1}\partial_x^2 c_1 - u_1 \partial_x c_1 + b\beta_1(c_2 - c_1), \quad (1.2a)$$

$$\partial_t c_2 = D_{x2}\partial_x^2 c_2 - u_2 \partial_x c_2 + b\beta_2(c_1 - c_2), \quad (1.2b)$$

where $\beta_1 = h/h_1 = q_1^{-1}$; $\beta_2 = h/h_2 = q_2^{-1}$.

Chikwendu (1986*b*) solved this system *exactly* and found that at large times the contaminant concentrations approached a Gaussian with a peak travelling at the mean velocity $\bar{u}$ and with dispersivity given by

$$D(2) = \frac{(q_1 q_2)^2 (u_1 - u_2)^2}{b} + q_1 D_{x1} + q_2 D_{z2}, \quad (1.3)$$

where q_1 and q_2 were the fractional thicknesses of the fast and slow zones respectively.

In this paper an exact integral expression is obtained for the dispersivity by extrapolating (1.3) to N zones and then taking the limit as $N \to \infty$. In this limit the discrete N-zone model becomes an exact continuum model. The resulting formula emphasizes the importance in dispersion of the velocity differences (shear) between the fast and slow regions of the flow. We begin with a three-zone model.

2. Three-zone model for two-dimensional channel flow

We consider steady laminar or turbulent two-dimensional flow of depth h in an open channel and for the purpose of analysis we regard the flow as being discretized into three layers (figure 2) with thicknesses h_1, h_2, h_3, concentrations c_1, c_2, c_3, velocities u_1, u_2, u_3, and diffusivities D_{x1}, D_{x2}, D_{x3} respectively. Let the contaminant mass flux between zone 1 and zone 2 be given by

$$M_{12} = d_{12}(c_2 - c_1), \quad (2.1a)$$

while the contaminant mass flux between zone 2 and zone 3 is

$$M_{23} = d_{23}(c_3 - c_2). \quad (2.1b)$$

Then the governing advective–diffusion equations are

$$\partial_t c_1 = D_{x1}\partial_x^2 c_1 - u_1 \partial_x c_1 + \frac{d_{12}}{h_1}(c_2 - c_1), \quad (2.2a)$$

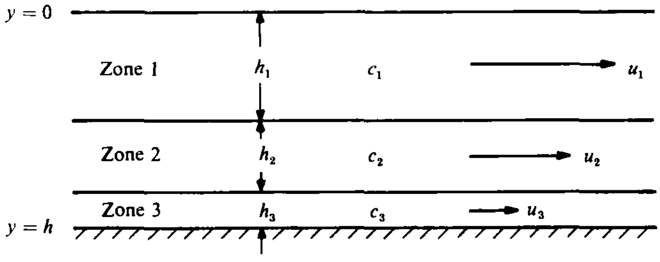

FIGURE 2. Three-zone model for open-channel flow.

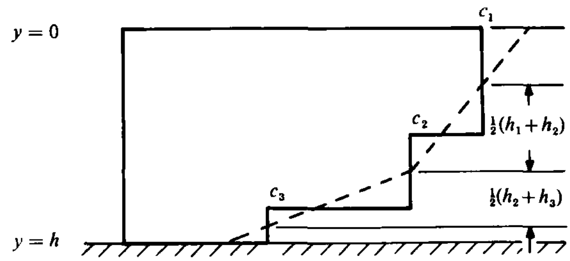

FIGURE 3. Concentration profile and mean gradients in the three-zone model.

$$\partial_t c_2 = D_{x2}\,\partial_x^2 c_2 - u_2\,\partial_x c_2 + \frac{d_{12}}{h_2}(c_1 - c_2) + \frac{d_{23}}{h_2}(c_3 - c_2), \tag{2.2b}$$

$$\partial_t c_3 = D_{x3}\,\partial_x^2 c_3 - u_3\,\partial_x c_3 + \frac{d_{23}}{h_3}(c_2 - c_3). \tag{2.2c}$$

By Fick's law, the contaminant mass flux across any horizontal layer is given by

$$\text{Flux} = -D_y(y)\frac{\partial c}{\partial y}. \tag{2.3}$$

Thus the mass flux between zones 1 and 2 can be approximated using

$$M_{12} \approx -D_{y12}\frac{(c_1 - c_2)}{\frac{1}{2}(h_1 + h_2)}, \tag{2.4}$$

where D_{y12} is the average vertical diffusivity between zones 1 and 2 and we have used the mean concentration gradient, which is shown in figure 3 as a dotted line from the midpoint of zone 1 to the midpoint of zone 2. By comparing (2.1a) and (2.4) we can calculate d_{12}. Similarly, we can calculate d_{23}, and using these results in (2.2) we find that the governing coupled dispersion equations for the three-zone model are

$$\partial_t c_1 = D_{x1}\,\partial_x^2 c_1 - u_1\,\partial_x c_1 + b_{12}\beta_1(c_2 - c_1), \tag{2.5a}$$

$$\partial_t c_2 = D_{x2}\,\partial_x^2 c_2 - u_2\,\partial_x c_2 + b_{12}\beta_2(c_1 - c_2) + b_{23}\beta_2(c_3 - c_2), \tag{2.5b}$$

$$\partial_t c_3 = D_{x3}\,\partial_x^2 c_3 - u_3\,\partial_x c_3 + b_{23}\beta_3(c_2 - c_3), \tag{2.5c}$$

where

$$\left.\begin{aligned} \beta_j &= \frac{h}{h_j} = q_j^{-1} \quad (j = 1, 2, 3), \\ b_{12} &= \frac{2D_{y12}}{h^2(q_1 + q_2)}; \quad b_{23} = \frac{2D_{y23}}{h^2(q_2 + q_3)}, \\ q_1 &+ q_2 + q_3 = 1, \end{aligned}\right\} \tag{2.6}$$

and D_{y23} is the average vertical diffusivity between zones 2 and 3.

By Fourier transformation using

$$F_j(\lambda, t) = \int_{-\infty}^{\infty} e^{i\lambda x} c_j(x, t)\, dx \quad (j = 1, 2, 3) \tag{2.7}$$

(2.5) can be written as a system of ordinary differential equations

$$\partial_t F_1 = -m_1(\lambda) F_1 + b_{12} \beta_1 F_1, \tag{2.8a}$$

$$\partial_t F_2 = -m_2(\lambda) F_2 + b_{12} \beta_2 F_1 + b_{23} \beta_2 F_3, \tag{2.8b}$$

$$\partial_t F_3 = -m_3(\lambda) F_3 + b_{23} \beta_3 F_2, \tag{2.8c}$$

where

$$m_1(\lambda) = \lambda^2 D_{x1} - i\lambda u_1 + b_{12} \beta_1, \tag{2.9a}$$

$$m_2(\lambda) = \lambda^2 D_{x2} - i\lambda u_2 + (b_{12} + b_{23}) \beta_2, \tag{2.9b}$$

$$m_3(\lambda) = \lambda^2 D_{x3} - i\lambda u_3 + b_{23} \beta_3. \tag{2.9c}$$

Assuming solutions of exponential form

$$F_j(\lambda, t) = P_j(\lambda)\, e^{\gamma t} \quad (j = 1, 2, 3) \tag{2.10}$$

we find that the characteristic equation for the system (2.8) is

$$\gamma^3 + (m_1 + m_2 + m_3)\gamma^2 + (m_1 m_2 + m_2 m_3 + m_3 m_1 - b_{12}^2 \beta_1 \beta_2 - b_{23}^2 \beta_2 \beta_3)\gamma + m_1 m_2 m_3 - b_{12}^2 \beta_1 \beta_2 m_3 - b_{23}^2 \beta_2 \beta_3 m_1 = 0. \tag{2.11}$$

This cubic equation for $\gamma(\lambda)$ is not easily solved and the resulting Fourier inversion integrals would be very difficult to evaluate. We can, however, determine the behaviour of the solution at large times from the behaviour of the Fourier transform when λ is small (see Chikwendu 1986*b*). Thus for small λ (large t) we neglect higher powers of λ and write

$$\gamma(\lambda) \sim B_0 + i\lambda B_1 - \lambda^2 B_2, \tag{2.12}$$

where B_0, B_1 and B_2 are constants. The Fourier inverse of (2.10) at large times is then

$$c_j(x, t) \sim \frac{P_j(0)}{2\pi} \int_{-\infty}^{\infty} \exp\left[B_0 t - i\lambda(x - B_1 t) - \lambda^2 B_2 t\right] d\lambda = \frac{P_j(0)}{(4\pi B_2 t)^{\frac{1}{2}}} \exp\left[\frac{-(x - B_1 t)^2}{4 B_2 t} + B_0 t\right]. \tag{2.13}$$

To calculate B_0, B_1 and B_2 we substitute (2.12) in (2.11) and equate the coefficients of λ^n to zero for $n = 0, 1, 2$, and we find that B_0 must satisfy

$$B_0^3 + \alpha_1 B_0^2 + \alpha_2 B_0 = 0, \tag{2.14}$$

where

$$\alpha_1 = b_{12}(\beta_1 + \beta_2) + b_{23}(\beta_2 + \beta_3), \tag{2.15a}$$

$$\alpha_2 = b_{12} b_{23}(\beta_1 \beta_2 + \beta_2 \beta_3 + \beta_3 \beta_1) = b_{12} b_{23} \beta_1 \beta_2 \beta_3. \tag{2.15b}$$

Thus the three solutions for B_0 are

$$B_{01} = 0, \tag{2.16a}$$

$$(B_{02}, B_{03}) = -\tfrac{1}{2}\alpha_1 \pm \tfrac{1}{2}(\alpha_1^2 - 4\alpha_2)^{\frac{1}{2}}. \tag{2.16b}$$

Both B_{02} and B_{03} are real and negative and in (2.13) would lead to exponential decay with time. We therefore require that $B_0 = 0$ in order to obtain the dominant behaviour of the contaminant concentration at large times, and we find, after some algebraic labour, that

$$B_1 = q_1 u_1 + q_2 u_2 + q_3 u_3 = \bar{u}, \tag{2.17a}$$

$$B_2 = \frac{q_1^2(q_2+q_3)^2}{b_{12}}(u_1-u_{23})^2 + \frac{(q_1+q_2)^2 q_3^2}{b_{23}}(u_{12}-u_3)^2 + q_1 D_{x1} + q_2 D_{x2} + q_3 D_{x3}, \tag{2.17b}$$

where $u_{12} = (q_1 u_1 + q_2 u_2)/(q_1+q_2)$ is the mean velocity in zones 1 and 2 combined, and $u_{23} = (q_2 u_2 + q_3 u_3)/(q_2+q_3)$ is the mean velocity in zones 2 and 3 combined.

It is clear from (2.13) that B_2 is the longitudinal dispersivity at large times and the contaminant concentrations approach the Gaussian

$$c_j(x,t) \sim c_0(4\pi Dt)^{-\frac{1}{2}} \exp\left[-\frac{(x-\bar{u}t)^2}{4Dt}\right] \quad (j = 1, 2, 3), \tag{2.18}$$

where c_0 depends on the initial contaminant mass discharged into the flow and the longitudinal dispersivity is given in (2.17*b*), i.e.

$$D(3) = \sum_{j=1}^{2} \frac{(q_1+q_2+\ldots+q_j)^2[1-(q_1+q_2+\ldots+q_j)]^2 [u_{12\ldots j} - u_{(j+1)\ldots 3}]^2}{b_{j(j+1)}} + \sum_{j=1}^{3} q_j D_{xj}. \tag{2.19}$$

3. N-zone model and the dispersivity in the limit as $N \to \infty$

If the two-dimensional open channel flow is divided into N zones of parallel flow with thicknesses h_j, contaminant concentrations c_j, average velocities u_j and average longitudinal diffusivities D_{xj} respectively, in each well mixed zone (with $j = 1, 2, \ldots, N$), then the N coupled dispersion equations are

$$\partial_t c_1 = D_{x1}\partial_x^2 c_1 - u_1 \partial_x c_1 + b_{12}\beta_1(c_2-c_1), \tag{3.1a}$$

$$\partial_t c_j = D_{xj}\partial_x^2 c_j - u_j \partial_x c_j + b_{(j-1)j}\beta_j(c_{j-1}-c_j) + b_{j(j+1)}\beta_j(c_{j+1}-c_j) \; (j = 2, 3, \ldots, N-1) \tag{3.1b}$$

$$\partial_t c_N = D_{xN}\partial_x^2 c_N - u_N \partial_x c_N + b_{(N-1)N}\beta_N(c_{N-1}-c_N), \tag{3.1c}$$

where

$$\beta_j = \frac{h}{h_j} = \frac{1}{q_j} \quad (j = 1, 2, \ldots, N), \tag{3.2a}$$

$$b_{j(j+1)} = \frac{2D_{yj(j+1)}}{h^2(q_j+q_{j+1})} \quad (j = 1, 2, \ldots, N-1), \tag{3.2b}$$

and $D_{yj(j+1)}$ is the average vertical diffusivity between zones j and $(j+1)$. The zonal longitudinal diffusivities D_{xj} are calculated in the Appendix.

In principle the system (3.1) can be analysed using the same approach as was used in the 3-zone case, i.e. Fourier transformation, the use of a large-time exponent $\gamma(\lambda)$ given by (2.12), and hence the calculation of $D(N)$. However, we can simply reason by induction from the 3-zone case, (2.19), that at large times the contaminant concentrations will be Gaussian (2.18) and the longitudinal dispersivity will be given by

$$D(N) = \sum_{j=1}^{N-1} (q_1+q_2+\ldots+q_j)^2 [1-(q_1+q_2+\ldots+q_j)]^2 \times [u_{12\ldots j} - u_{(j+1)\ldots N}]^2 / b_{j(j+1)} + \sum_{j=1}^{N} q_j D_{xj}, \tag{3.3}$$

where

$$u_{12\ldots j} = \left(\sum_{k=1}^{j} q_k u_k\right) \Big/ \left(\sum_{k=1}^{j} q_k\right) = \text{mean velocity in the first } j \text{ zones,} \qquad (3.4a)$$

$$u_{(j+1)\ldots N} = \left(\sum_{k=j+1}^{N} q_k u_k\right) \Big/ \left(\sum_{k=j+1}^{N} q_k\right) = \text{mean velocity in the last } (N-j) \text{ zones.} \qquad (3.4b)$$

3.1. *The limit as $N \to \infty$*

As the number of zones tends to infinity the zone thicknesses become infinitesimal and our sums become integrals. If $q = y/h$ represents dimensionless depth in the flow, then as $N \to \infty$,

$$q_j \to \Delta q; \quad q_1 + q_2 \ldots q_j \to \int_0^q \mathrm{d}q = q, \qquad (3.5a)$$

$$b_{j(j+1)} \to \frac{D_y(q)}{h^2 \Delta q}; \quad \sum_{j=1}^{N} q_j D_{xj} \to \int_0^1 D_x(q)\,\mathrm{d}q, \qquad (3.5b)$$

$$u_{12\ldots j} \to \frac{1}{q}\int_0^q u(q')\,\mathrm{d}q' = u_{\mathrm{f}}(q) = \text{average velocity in the faster layer of a two-layer model in which the fractional thicknesses are } q \text{ and } (1-q), \qquad (3.5c)$$

$$u_{(j+1)\ldots N} \to \frac{1}{(1-q)}\int_q^1 u(q')\,\mathrm{d}q' = u_{\mathrm{s}}(q) = \text{average velocity in the slower layer of a two-layer model in which the fractional thicknesses are } q \text{ and } (1-q). \qquad (3.5d)$$

Thus in the limit as $N \to \infty$ the longitudinal dispersivity at large times is given by

$$D(\infty) = \lim_{N \to \infty} D(N) = h^2 \int_0^1 \frac{q^2(1-q)^2}{D_y(q)} [u_{\mathrm{f}}(q) - u_{\mathrm{s}}(q)]^2\,\mathrm{d}q + \int_0^1 D_x(q)\,\mathrm{d}q. \qquad (3.6)$$

It may be noted that the N-zone model is an approximation of the actual physical situation, but in the limit as $N \to \infty$ the model becomes exact.

3.2. *Plane Poiseuille flow*

For laminar flow between parallel plates with separation $2h$ the velocity profile is

$$u(q) = \tfrac{3}{2}\bar{u}(1-q^2), \qquad (3.7)$$

and from (3.5*c*, *d*) the fast- and slow-zone average velocities are

$$u_{\mathrm{f}}(q) = \tfrac{3}{2}\bar{u}\left(1 - \frac{q^2}{3}\right), \quad u_{\mathrm{s}}(q) = \tfrac{1}{2}\bar{u}(1-q)(2+q). \qquad (3.8a, b)$$

The diffusivities are $D_y(q) = D_x(q) = k$, the molecular diffusivity. Using these velocities in (3.6), we find that

$$D(\infty) = \frac{2h^2\bar{u}^2}{105k} + k, \qquad (3.9)$$

which is the result obtained by using Taylor's (1953) method.

3.3. *Turbulent open-channel flow*

For turbulent channel flow we can use the logarithmic velocity profile

$$u(q) = \bar{u}+\frac{u_*}{\kappa}[1+\log(1-q)], \tag{3.10}$$

where u_* is the friction velocity and κ is the von Kármán constant. The fast- and slow-zone average velocities are

$$u_f(q) = \bar{u}-\frac{(1-q)\,u_*}{q\kappa}\log(1-q), \quad u_s(q) = \bar{u}+\frac{u_*}{\kappa}\log(1-q), \tag{3.11a, b}$$

and the diffusivity is (Elder 1959)

$$D_y(q) = D_x(q) = h\kappa u_* q(1-q). \tag{3.12}$$

Thus, from (3.6),

$$D(\infty) = \frac{hu_*}{\kappa^3}\int_0^1 q^{-1}(1-q)[\log(1-q)]^2\,dq+\tfrac{1}{6}\kappa hu_*$$

$$= \frac{0.4041}{\kappa^3}hu_*+\tfrac{1}{6}\kappa hu_*, \tag{3.13}$$

again the same as Elder's result.

4. Pipe flow

In the case of contaminant dispersion in laminar or turbulent flow in a circular pipe of radius a the flow can be discretized into N zones of parallel flow, where the jth zone is the annular region between concentric circles of radii r_{j-1} and $r_j (j = 1, 2, \ldots, N)$, with $r_0 = 0$ and $r_N = a$, as in figure 4.

If A is the pipe cross-sectional area and A_j the cross-sectional area of the jth zone, then the fractional area of the jth zone is

$$q_j = \frac{A_j}{A} = \frac{r_j^2-r_{j-1}^2}{a^2} = \frac{1}{\beta_j}. \tag{4.1}$$

We can define

$$p_j = \frac{r_j}{a} = \text{dimensionless radius of the } j\text{th circle},$$

$$W_j = p_j-p_{j-1} = \text{dimensionless radial width of the } j\text{th zone}.$$

Then,

$$q_j = W_j(p_j+p_{j-1}). \tag{4.2}$$

The mass flux between zone j and zone $(j+1)$ can be written as

$$M_{j(j+1)} = d_{j(j+1)}\,(c_{j+1}-c_j),$$

and the contact length between these two zones is

$$L_j = 2\pi r_j.$$

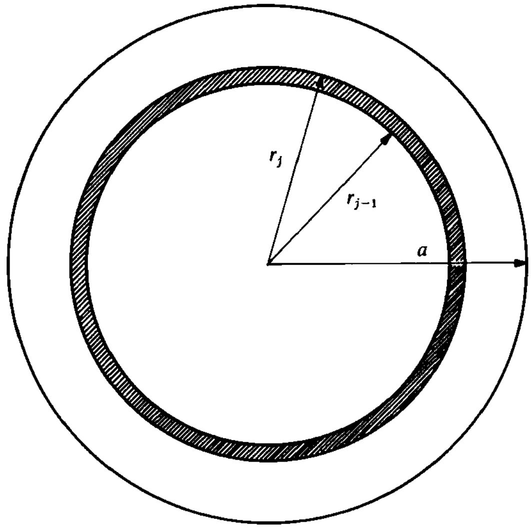

FIGURE 4. Pipe cross-section showing the jth zone.

Thus the coupled advective–diffusion equations are

$$\partial_t c_1 = D_{x1}\partial_x^2 c_1 - u_1 \partial_x c_1 + \frac{L_1 d_{12}}{A_1}(c_2 - c_1), \tag{4.3a}$$

$$\partial_t c_j = D_{xj}\partial_x^2 c_j - u_j \partial_x c_j + \frac{L_{j-1} d_{(j-1)j}}{A_j}(c_{j-1} - c_j) + \frac{L_j d_{j(j+1)}}{A_j}(c_{j+1} - c_j) \quad (j = 2, 3, \ldots, N-1) \tag{4.3b}$$

$$\partial_t c_N = D_{xN}\partial_x^2 c_N - u_N \partial_x c_N + \frac{L_{N-1} d_{(N-1)N}}{A_N}(c_{N-1} - c_N). \tag{4.3c}$$

By Fick's law the mass flux is given by

$$M_{j(j+1)} = -d_{j(j+1)}(c_{j+1} - c_j) \approx -\frac{2D_{rj(j+1)}(c_{j+1} - c_j)}{(W_j + W_{j+1})a}, \tag{4.4}$$

where $D_{rj(j+1)}$ is the average radial diffusivity between zones j and $(j+1)$, and we have used the average concentration gradient between these zones.

We can calculate $d_{j(j+1)}$ from (4.4) and use this in (4.3). It is thus found that the N coupled dispersion equations are again given by (3.1), where in this case the β_j are defined in (4.1) and

$$b_{j(j+1)} = \frac{4p_j D_{rj(j+1)}}{(W_j + W_{j+1})a^2} \quad (j = 1, 2, \ldots, N-1). \tag{4.5}$$

The dispersivity at large times is again given by (3.3), (3.4) with q_j as defined in (4.2). Since $q_1 + q_2 + \ldots + q_j = p_j^2$, this dispersivity can be written as

$$D(N) = \sum_{j=1}^{N-1} \frac{p_j^4(1-p_j^2)^2 [u_{12\ldots j} - u_{(j+1)\ldots N}]^2}{b_{j(j+1)}} + \sum_{j=1}^{N} q_j D_{xj}. \tag{4.6}$$

4.1. *The limit* $N \to \infty$

For pipe flow, as $N \to \infty$ we again go for the discrete model to the continuum limit, and if $p = r/a$, then

$$W_j = p_j - p_{j-1} \to \Delta p, \quad p_j = \sum_{k=1}^{j} W_k \to \int_0^p \mathrm{d}p = p, \tag{4.7a, b}$$

$$q_j = W_j(p_j + p_{j-1}) \to 2p\Delta p, \quad \sum_{k=1}^{j} q_k = p_j^2 \to p^2, \quad b_{(j-1)j} \to \frac{2pD_r(p)}{a^2 \Delta p}, \tag{4.7c, d, e}$$

$$u_{12\ldots j} = \frac{1}{p_j^2} \sum_{k=1}^{j} q_k u_k \to \frac{1}{p^2} \int_0^p 2p' u(p') = u_{\mathrm{f}}(p), \tag{4.7f}$$

$$u_{(j+1)\ldots N} = \frac{1}{1-p_j^2} \sum_{k=j+1}^{N} q_k u_k \to \frac{1}{1-p^2} \int_p^1 2p' u(p')\, \mathrm{d}p' = u_{\mathrm{s}}(p). \tag{4.7g}$$

Thus, from (4.6) and (4.7) the large-time dispersivity in laminar or turbulent pipe flow is found to be

$$D(\infty) = \lim_{N \to \infty} D(N) = a^2 \int_0^1 [2pD_r(p)]^{-1} p^4 (1-p^2)^2 [u_{\mathrm{f}}(p) - u_{\mathrm{s}}(p)]^2 \, \mathrm{d}p + \int_0^1 2pD_x(p)\, \mathrm{d}p. \tag{4.8}$$

4.2. *Poiseuille pipe flow*

For laminar pipe flow the velocity profile is

$$u(p) = 2\bar{u}(1-p^2), \tag{4.9}$$

and the average fast- and slow-zone velocities are respectively

$$u_{\mathrm{f}}(p) = \bar{u}(2-p), \quad u_{\mathrm{s}}(p) = \bar{u}(1-p). \tag{4.10a, b}$$

Thus, $u_{\mathrm{f}}(p) - u_{\mathrm{s}}(p) = \bar{u}$, and since $D_x(p) = D_r(p) = k$, the molecular diffusivity, the large-time dispersivity is, from (4.8),

$$D(\infty) = \frac{a^2 \bar{u}^2}{k} \frac{1}{2} \int_0^1 p^3 (1-p^2)^2 \, \mathrm{d}p + k$$

$$= \frac{a^2 \bar{u}}{48k} + k, \tag{4.11}$$

which is Taylor's result.

5. Conclusion

The expressions (3.6) and (4.8) derived in this paper for the large-time longitudinal dispersivity in shear flows emphasize the significance of the velocity differences between the fast and slow regions of the flow, i.e. $u_{\mathrm{f}} - u_{\mathrm{s}}$. In fact the overall shear dispersivity is the integral sum of the dispersivities obtained from all possible choices of fast- and slow-zone thicknesses.

However, these formulae can also be expressed in terms of the velocity deviation from the mean, i.e. $u' = u - \bar{u}$, in the manner of Taylor. Thus using (3.5*c*, *d*) we can write

$$q(1-q)[u_{\mathrm{f}}(q) - u_{\mathrm{s}}(q)] = \int_0^q u'(q')\, \mathrm{d}q', \tag{5.1}$$

where $$u'(q) = u(q) - \bar{u}.$$

Similarly, using (4.7*f*, *g*), we can write

$$p^2(1-p^2)\,[u_f(p)-u_s(p)] = \int_0^p 2p'u'(p')\,dp'. \tag{5.2}$$

Thus from (3.6) and (4.8) the large-time shear dispersivity in channel and pipe flow respectively can be expressed as

$$D(\infty) = h^2 \int_0^1 D_y^{-1}(q)\left[\int_0^q u'(q')\,dq'\right]^2 dq + \int_0^1 D_x(q)\,dq, \tag{5.3a}$$

and $$D(\infty) = a^2 \int_0^1 [2pD_r(p)]^{-1}\left[\int_0^p 2p'u'(p')\,dp'\right]^2 dp + \int_0^1 2pD_x(p)\,dp, \tag{5.3b}$$

These equations are in the form of Elder's (1959) formula, and are of course completely equivalent to the result of Aris (1956, equation 40).

Finally, it may be noted that the zonal approach presented in this paper may be useful in the study of other dispersion problems and other transport phenomena.

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Appendix. Two-stage description and the calculation of the zonal longitudinal diffusivities D_{xj}

In order to calculate the zonal longitudinal diffusivities D_{xj}, which appear in (3.1), we regard the dispersion process *as if it takes place in two stages* following contaminant discharge into the flow.

In the *first stage* the contaminant mass in each zone disperses exclusively in that zone with no contaminant-mass exchange between the zones. After some time the Taylor asymptotic dispersion stage would be reached in each zone and the governing advective–diffusion equations would be separate one-dimensional dispersion equations in each zone, that is,

$$\partial_t c_j = D_{xj}\,\partial_x^2 c_j - u_j\,\partial_x c_j \quad (j = 1, 2, \ldots, N). \tag{A 1}$$

Thus D_{xj} is defined as the Taylor longitudinal dispersivity that would be obtained in the jth zone if there were no contaminant mixing between the zones. The solution of (A 1) would lead to N separate Gaussian clouds moving downstream.

In the *second stage* of the dispersion process contaminant-mass exchange takes place between the zones. This exchange is modelled in §3 of this paper and leads to (3.1). Thus at large times a single Gaussian cloud is obtained.

It should be pointed out that the two-stage description given here is an extension of Sullivan's (1971) three-stage description of turbulent dispersion. Of course in turbulent flow there is a third stage in which contaminant mixing takes place between the mainstream and the viscous sublayer, but in laminar dispersion the final asymptotic dispersion state is reached in the second stage and there is no third stage (see Chikwendu 1986*a*).

Zonal longitudinal diffusivities D_{xj} in plane Poiseuille flow

For laminar flow of depth h, consider the jth zone whose dimensionless thickness is $q_j = h_j/h$ and which lies in the layer $Y_j \leqslant q \leqslant Y_j + q_j$, where $q = y/h$ is the dimensionless vertical distance downward from the maximum velocity surface of the flow and

$$Y_j = \sum_{k=1}^{j-1} q_k, \quad Y_1 = 0, \quad Y_{N+1} = 1.$$

The mean velocity in this zone is thus

$$u_j = \frac{1}{q_j}\int_{Y_j}^{Y_j+q_j} u(q)\,\mathrm{d}q = \tfrac{3}{2}\bar{u}(1 - Y_j q_j - Y_j^2 - \tfrac{1}{3}q_j^2), \tag{A 2}$$

where $u(q)$ is given in (3.7).

We define

$$z = \frac{q-Y_j}{q_j} = \text{depth measured from the upper surface of the } j\text{th zone, divided by the zonal thickness.} \tag{A 3}$$

If the jth zone were divided into two layers as follows,

(i) a faster layer in which $Y_j \leqslant q \leqslant Y_j + zq_j$ (thickness zq_j) with $0 \leqslant z \leqslant 1$,

(ii) a slower layer in which $Y_j + zq_j \leqslant q \leqslant Y_j + q_j$ (thickness $q_j - zq_j$),

then, by applying (A 2) we find that the mean velocities in these two layers would be, respectively,

$$u_{\mathrm{f}j} = \tfrac{3}{2}\bar{u}[1 - Y_j(zq_j) - Y_j^2 - \tfrac{1}{3}(zq_j)^2] \tag{A 4a}$$

$$u_{\mathrm{s}j} = \tfrac{3}{2}\bar{u}[1 - (Y_j + zq_j)(q_j - zq_j) - (Y_j + zq_j)^2 - \tfrac{1}{3}(q_j - zq_j)^2]. \tag{A 4b}$$

Thus, from (3.6), the Taylor dispersivity in the jth zone is given by

$$D_{xj} = (hq_j)^2 \int_0^1 \frac{z^2}{k}(1-z)^2\,[u_{\mathrm{f}j}(z) - u_{\mathrm{s}j}(z)]^2\,\mathrm{d}z + \int_0^1 k\,\mathrm{d}z. \tag{A 5}$$

By using (A 4) in (A 5) and integrating, it is found that in laminar two-dimensional flow, the longitudinal diffusivity in the jth zone is given by

$$D_{xj} = \frac{\bar{u}^2 h^2}{k}\left(\tfrac{3}{40} Y_j^2 q_j^4 + \tfrac{3}{40} Y_j q_j^5 + \tfrac{2}{105} q_j^6\right) + k. \tag{A 6}$$

It may be noted that as the number of zones tends to infinity the zonal thicknesses q_j approach zero and D_{xj} approaches k, the molecular diffusivity.

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